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GENERALIZED PARAMETRIC OSCILLATIONS OF MECHANICAL SYSTEMS†

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An effective numerical–analytical method of investigating parametrically excited oscillatory Hill-type systems, described by general boundary-value problems, is developed. It is assumed that the coefficients of the equation depend in an arbitrary non-linear way on a parameter, the eigenvalues of which are to be obtained. The approach to solving the generalized periodic boundary-value problem is based on the established differential relation between the eigenvalue and the value of the period (the length of the interval). The computational algorithm possesses the property of accelerated convergence, which enables many extremely subtle and difficult problems of constructing the dependences of the eigenvalues and eigenfunctions (the forms with the oscillations) on the index and parameters of the system, difficult to obtain by traditional approaches, to be successfully investigated. To illustrate the high efficiency of the method, a solution of the problem of the spatial angular oscillations of a dynamically symmetrical artificial satellite moving in a circular orbit is constructed.

1. FORMULATION OF THE PROBLEM

Consider the oscillations of a linear system, described by the generalized periodic boundary-value problem for a Hill-type equation [1]

$$\ddot{u}+r(t,\lambda)u=0,\quad u(0)=u(1),\quad \dot{u}(0)=\dot{u}(1)$$
$$0\leqslant t\leqslant 1,\quad \lambda\in\Lambda\quad (r(t+1,\lambda)\equiv r(t,\lambda)) \tag{1.1}$$

Here r is a piecewise-continuous real function of the argument t, which can be continued in a similar (piecewise-continuous) way for all $|t|\geqslant 0$. In particular, it can be continuous in t. It is assumed to be continuous with respect to the real parameter λ of the function r, and continuously differentiable when $\lambda\in\Lambda$; Λ is a set (a set of segments) and admissible values of the required parameter Λ.

It is required to obtain the solution of the generalized problem (1.1), i.e. to determine the eigenvalues λ and the functions $u(t,\lambda)$. The case of a discrete spectrum $\lambda\in\{\lambda_n\}$ and the set of modes of oscillation $\{u_n(x)\}$, $n=1,2,\ldots$ are of particular interest for applications. The coefficient r also usually depends on other parameters of the system λ (geometrical, inertial, elastic, etc.), i.e. $r=r(t,\lambda,\gamma)$, $\gamma\in\Gamma$. It is required to construct curves or surfaces $\lambda_n(\gamma)$ and the modes of oscillation $u_n(t,\gamma)$ corresponding to them (see below).

Remarks. 1. In the case of the classical formulation of the periodic problem the function r is linear in λ; it is usually assumed that $r\equiv\lambda+\gamma q(t)$ (Hill's equation), in particular $q\equiv\cos 2\pi t$ (Mathieu's equation) or $q\equiv\operatorname{sign}(\cos 2\pi t)$ (Meissner's equation); it is required to obtain $\lambda_n(\gamma)$, $u_n(t,\gamma)$ (see [1–6]).

2. A more general form of the equation $a\ddot{u}+b\dot{u}+cu=0$, where a, b and c are periodic functions of t and depend on the parameters λ and γ, with corresponding assumptions regarding smoothness and sign definiteness, can be reduced to the form (1.1) [1–6]. Similarly, a system of two first-order equations with periodic coefficients (for example, the linear Hamilton equations) can be represented in the form (1.1). Situations when this reduction is difficult are also of interest for applied problems.

3. In addition to the main 1/1 resonance and the $1/n$ resonances, the $2/n$ resonances can occur in system (1.1) and, possibly, also more complex combination resonances [5]. The determination of the corresponding values of $\lambda_n(\gamma)$ and the periodic solutions $u_n(t,\gamma)$ is of particular interest and is the purpose of the following investigations.

4. In formulations of periodic boundary-value problems (unlike the Sturm–Liouville problem) the function r is not usually assumed to be positive definite for the values of t, λ and γ under consideration. This leads to substantial difficulties in using oscillation theorems and Sturm's comparison theorems [4] and considerable complications when constructing solutions and analysing them.

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5. An investigation of the generalized periodic problem described by the Hill-type equation (1.1) is important for the theory of non-linear oscillations and stability [3, 5–7]. These problems arise in mechanics, applied celestial mechanics, electromechanics, hydrodynamics, the theory of elasticity, electrodynamics and atomic physics, etc. The main analytical and numerical results have been obtained for the classical cases of Hill's equation: Mathieu's and Meissner's equations [1–8]. There are no effective numerical–analytical methods and results of investigations of the generalized periodic boundary-value problems, due to the fundamental difficulties in solving them.

6. The main properties and behaviour of the eigenvalues $\lambda_n(\gamma)$ and the functions $u_n(t,\gamma)$ of the generalized problem may differ essentially from the "standard" ones, well known for the classical problems mentioned above [1–9]. We will illustrate this assertion using simple examples for an Euler-type equation, for which a complete analytical solution can be obtained [4]

$$\begin{gathered}
r(t,\lambda)=(\lambda+t)^{-2},\quad \lambda\in\Lambda=\{\lambda:\lambda>0,\ \lambda<-1\},\quad 0\leqslant t\leqslant 1\\
\lambda_n=\operatorname*{Arg}_{\lambda}\Big[\cos\big(\sqrt{3}\ln\chi\big)-\sqrt{3}(\chi^2-1)(\chi^2+1)^{-1}\sin\big(\sqrt{3}\ln\chi\big)-\\
-2\chi(\chi^2+1)^{-1}\Big],\quad \lambda_n\approx\Big(\exp\big[2\pi(n+1/3)/\sqrt{3}\big]-1\Big)^{-1}\\
u_n(t)=c_n\xi_n(t)[\chi_n\sin\theta_n(1)\cos\theta_n(t)+(1-\chi_n\cos\theta_n(1))\sin\theta_n(t)]\\
\xi_n(t)=(1+t/\lambda_n)^{1/2},\quad \chi_n=\xi_n(1),\quad \theta_n(t)=\sqrt{3}\ln\xi_n(t),\quad n=\pm1,\pm2,\ldots
\end{gathered}\tag{1.2}$$

Here c_n are arbitrary constants. It follows from (1.2) that the eigenvalues λ_n are concentrated in the region of $\lambda=+0$ ($n\geqslant 1$) and $\lambda=-1-0$ ($n\leqslant -1$), and they tend exponentially to these values as $|n|\to\infty$. The eigenfunctions $u_n(t)$ (1.2) oscillate with respect to r when $|n|$ is fairly large as rapidly as desired and, moreover, in an exotic way.

The example when the function $r=\lambda^2(1+\lambda t)^{-2}$, where $\lambda>-1$, can be investigated in a similar way. In expressions (1.2) one should make the replacement $\lambda\to\lambda^{-1}$; we obtain the following approximate expressions for λ_n

$$\lambda_n\approx\exp[2\pi(n+1/3)/\sqrt{3}]-1,\quad n=\pm1,\pm2,\ldots\tag{1.3}$$

According to (1.2) and (1.3) $\lambda_n\to+\infty$ as $n\to+\infty$ and $\lambda_n\to\lambda{-1}$ as $n\to-\infty$. The corresponding eigenfunctions for large $|n|$ also oscillate rapidly.

Other examples can be considered which illustrate various unusual properties of the eigenvalues and eigenfunctions. The spectrum of generalized periodic boundary-value problem (1.1) may be discrete (denumerable or finite), discrete-continuous or empty. In this case the discrete values of the spectrum can be both simple and multiple (for example, when $r=\text{const}$), to which one or two eigenfunctions correspond.

7. Generalized problem (1.1) belongs to the class of non-self-conjugate operators, unlike the corresponding classical case. Such problems have a number of specific features, one of which is the form of the orthogonality condition

$$\int_0^1[r(t,\lambda_n)-r(t,\lambda_m)]u_n(t)u_m(t)dt=0,\quad r(t,\lambda_n)\not\equiv r(t,\lambda_m)\tag{1.4}$$

For our further constructions we will need to know the norms of the functions $u_n(t)$ with a certain weight. Taking the limit in (1.4) as $\lambda_m\to\lambda_n$ we obtain the expression (see below)

$$N_n=\int_0^1 r'_\lambda(t,\lambda_n)u_n^2(t)dt\tag{1.5}$$

Under additional conditions of sign-definiteness, imposed on the derivative r'_λ with respect to λ, the expression for N_n (1.5) has the meaning of the square of the norm of the function $u_n(t)$ with weight $r'_\lambda(t,\lambda_n)$. When the function r is linear in λ; relations (1.4) and (1.5) are identical with those commonly known for self-conjugate boundary-value problems [1, 2, 4].

Using a formal method, problem (1.1) can be represented in the form of a continuous family of classical problems, in which $\lambda\in\Lambda$ is a parameter of the family. In fact, we will consider a problem for μ and U of the form

$$\begin{gathered}
\ddot U+[\mu+r(t,\lambda)]U=0,\quad 0\leqslant t\leqslant 1,\quad U(0)=U(1),\quad \dot U(0)=\dot U(1)\\
\mu\in\{\mu_m(\lambda)\},\quad U_m(t,\lambda)=U(t,\mu_m(\lambda),\lambda),\quad \lambda\in\Lambda,\quad m=1,2,\ldots
\end{gathered}\tag{1.6}$$

Suppose a fairly complete family of solutions of the auxiliary problem is constructed in accordance with (1.6). Then, the required solution of original problem (1.1) is obtained by carrying out the following operations

$$\lambda_n = \operatorname{Arg}\mu_m(\lambda),\quad \lambda\in\Lambda\quad (\mu_m(\lambda_n)=0,\quad n=n(m))$$
$$u_n(t) = U(t,0,\lambda_n) \tag{1.7}$$

This approach involves constructing a discrete set of functions $\mu_m(\lambda)$ and determining their real zeros λ_n. These constructions, as a rule, can only be carried out numerically for a comparatively small set of values of m (for the lower modes of oscillation) and corresponding $n(m)$, which may require extremely expensive software. Analytically complete calculations can be carried out in the extremely rare cases when Eqs (1.1) or (1.6) are integrable in closed form (see examples (1.2) and (1.3)). An approach based on relations (1.6) and (1.7) is, however, useful at the preliminary stage of investigating problems of the existence of the required solution, since self-conjugate problem (1.6) allows the use of variational methods [3, 4]. This enables us to obtain effective upper limits of $\mu_m(\lambda)$, for example, using the Rayleigh–Ritz method. We can also conveniently apply the well-developed shooting and finite-element methods to problem (1.6) in combination with the procedure of continuation with respect to the parameter λ to construct rough estimates.

2. THE METHOD OF ACCELERATED CONVERGENCE FOR SOLVING THE AUXILIARY PROBLEM

We will briefly describe the method of accelerated convergence, based on the introduction of a small parameter and the Lyapunov–Poincaré perturbation method [5]. We will assume that, for a certain fixed value of $\lambda\in\Lambda$ an estimate $\mu_m^0(\lambda)$ is obtained, for example, using the variational approach (the Rayleigh–Ritz method or Rayleigh's principle), the finite-element method, the shooting method, etc. In a number of cases it is possible to obtain a value $\lambda\in\Lambda$ for which an exact or approximate analytical solution is known (for example, as in the case of Mathieu's and Meissner's equations). For this $\mu_m^0(\lambda)$ we construct a general solution V_{0m} of Eq. (1.6) (the subscript m is fixed, so we will henceforth omit it for brevity)

$$\ddot V + [\mu^0(\lambda) + r(t,\lambda)]V = 0,\quad V_0(t,\lambda) = C_1V_1^0(t,\lambda) + C_2V_2^0(t,\lambda)$$
$$1)\ V(0)=0,\quad \dot V(0)=1;\quad 2)\ V(0)=1,\quad \dot V(0)=0,\quad C_{1,2}=\text{const} \tag{2.1}$$

Here V_1^0 and V_2^0 are particular linearly independent solutions of the two Cauchy problems 1 and 2, which for known μ^0 with chosen $\lambda\in\Lambda$ can be constructed analytically or numerically, i.e. in the form of a computational procedure.

We will require that for a certain $t=\theta>0$ conditions of type (1.6) are satisfied: $V_0(0) = V_0(\theta)$, $\dot V_0(0) = \dot V_0(\theta)$. The following relation for θ follows from these boundary conditions, taking Liouville's theorem into account

$$\Delta^0(\theta,\lambda)=0,\quad \Delta^0(t,\lambda)\equiv \dot V_1^0(t,\lambda)+V_2^0(t,\lambda)-2 \tag{2.2}$$

Here Δ^0 is the determinant of the matrix of the coefficients for C_1 and C_2. Note that the Wronskian is constant and equal to unity.

We will further consider (2.2) as the equation for the unknown θ (λ is fixed). When $\mu^0=\mu$ (μ is the exact value), a root $\theta=1$ exists, and, in view of the continuous dependence, for μ^0 sufficiently close to μ a root θ^0 exists close to $\theta=1$. The quantity θ^0 can be determined effectively by numerical or analytical integration of Cauchy problems (2.1) and by calculating the determinant $\Lambda^0(t,\lambda)$ using (2.2).

It follows from the above that we can introduce as a natural measure of closeness of the initial approximation of the quantity μ^0 to μ a quantity ε

$$\varepsilon = 1-\theta^0,\quad |\varepsilon|\ll 1,\quad \theta^0 = \arg\min_j |1-\theta_j^0|,\quad \theta_j^0(\lambda) = \operatorname{Arg}\Delta^0(\theta,\lambda) \tag{2.3}$$

It is reasonable to seek the roots θ_j^0 (2.3) of the equation $\Lambda^0=0$ (2.2) in the neighbourhood of $\theta=1$, which implies the smooth continuability of the function r to $\theta>1$. The properties of the function $\Delta^0(t,\lambda)$ are established by using numerical experiment. Note that when $\mu^0+r>0$ the required quantity θ^0 is the mth positive root. However, in applied problems the function r is sign-variable and, moreover, the required λ corresponds to $\mu=0$. This makes it difficult to use the conclusions from Sturm's comparison theorems [4] and requires a thorough numerical investigation of the function Δ^0 when determining θ^0 using (2.3).

Employing the methods of perturbation theory [3–5, 9], we will consider the problem of the local behaviour of μ when the length of the interval $0 \leqslant t \leqslant \theta$ changes in the neighbourhood of the value $\theta = 1$. Using the approach described in [9] we obtain (λ is a fixed number)

$$\begin{aligned} &\mu = \mu^0 + \varepsilon M^0 + O(\varepsilon^2) \\ &M^0 = -(\dot V_0^2(\theta^0,\lambda) + (\mu^0(\lambda) + r(\theta^0,\lambda))V_0^2(\theta^0,\lambda)) \| V_0 \|^{-2} \\ &\lim_{\varepsilon\to 0} \frac{\mu-\mu^0}{\varepsilon} = \mu'_\theta(\lambda)\Big|_{\theta=1} = -(\dot U^2(1,\lambda) + (\mu(\lambda)+r(1,\lambda))U^2(1,\lambda)) \| U \|^{-2} \end{aligned} \tag{2.4}$$

Here $\|V^0\|^2$ and $\|U^0\|^2$ are the squares of the norm of the functions V_0 and U, defined in the usual way. It follows from (2.4) that for the case $M^0 \neq 0$ the following inequalities hold

$$|\varepsilon| \leqslant C_\mu |\mu-\mu^0|, \quad |\mu-\mu^0| \leqslant C_\varepsilon |\varepsilon|, \quad C_{\mu,\varepsilon} > 0 \tag{2.5}$$

We will further assume that the required value of θ^0 has been obtained. If $M^0 \neq 0$, this value is simple, i.e. $\Delta^{0\prime}(\theta^0(\lambda), \lambda) \neq 0$. The rank of the matrix mentioned above is then equal to unity, and the required solution V_0 (2.1) can be represented in two equivalent forms

$$\begin{aligned} V_0(t,\lambda) &= C_1[V_1(t,\lambda) + (1-\dot V_1(\theta^0,\lambda))\dot V_2^{-1}(\theta^0,\lambda)V_2(t,\lambda)] = \\ &= C_2[(1-V_2(\theta^0,\lambda))V_1^{-1}(\theta^0,\lambda)V_1(t,\lambda) + V_2(t,\lambda)] \end{aligned} \tag{2.6}$$

Here $C_{1,2}$ are arbitrary constants, which can be chosen from the normalization condition $\|V^0\|^2 = 1$. Both expressions (2.6) satisfy mixed boundary conditions of periodicity in the interval $0 \leqslant t \leqslant \theta^0$, which is established directly taking into account the definition of θ^0 (2.3).

In the critical case when θ^0 is a multiple root, i.e. $\Delta^{0\prime}(\theta^0(\lambda), \lambda) = 0$, the rank of the matrix is equal to unity or zero. In the second case the solution V_0 (2.1), which satisfies the boundary conditions, contains two arbitrary constants C_1 and C_2. This situation occurs, in particular, when $r = \text{const}$ (Mathieu's, Meissner's or Hill's equations with $\delta = 0$) and, possibly, in other cases of equations equivalent to an equation with constant coefficients. For the function V_0 one of the solutions is substituted into (2.4) (for $C_1 = 0$ or $C_2 = 0$); in the general case splitting of the eigenvalue occurs.

We will consider solution (2.1)–(2.3), (2.6) of the boundary-value problem that has been constructed in the interval $0 \leqslant t \leqslant \theta^0$ as an approximate solution of the original problem (1.6). We will use it to calculate a refined value of μ, with an error $O(\varepsilon^2)$ from (2.4)

$$\begin{aligned} &\mu^{(1)}(\lambda) = \mu^0(\lambda) + \varepsilon M^0(\lambda), \quad |\mu^{(1)}(\lambda) - \mu(\lambda)| \leqslant C_\varepsilon \varepsilon^2 \\ &M^0(\lambda) = -\left[\dot V_0^2(\theta^0,\lambda) + (\mu^0(\lambda) + r(\theta^0,\lambda))V_0^2(\theta^0,\lambda)\right] \| V_0 \|^{-2} \end{aligned} \tag{2.7}$$

Without loss of accuracy in powers of the small parameter ε in the expression for $M^0(\lambda)$ (2.7) we can put $\theta^0 = 1$, since $1 - \theta^0 = \varepsilon$. On the basis of the refined value of $\mu^{(1)}(\lambda)$ a general solution $V_0^{(1)}$ of Cauchy problem (2.1) is constructed, and from conditions (2.2) and (2.3) we find the quantity $\theta^{(1)}$, which determines the quantity $\varepsilon^{(1)}$: $\varepsilon^{(1)} = 1 - \theta^{(1)}(\lambda)$, where, according to (2.5) and (2.7), $|\varepsilon^{(1)}| \leqslant C_\mu C_\varepsilon \varepsilon^2$. The solution just constructed is assumed to be approximate for the original problem (1.6) with accuracy $O(\varepsilon^2)$. This simple procedure for refining the solution can be continued in the form of a recursive algorithm [10]

$$\begin{aligned} &\mu^{(k+1)}(\lambda) = \mu^{(k)}(\lambda) + \varepsilon^{(k)} M^{(k)}(\lambda), \quad \varepsilon^{(k)} = 1 - \theta^{(k)}(\lambda), \quad k = 0,1,2,\ldots \\ &M^{(k)}(\lambda) = -[\dot V_0^{(k)2}(\theta^{(k)},\lambda) + (\mu^{(k)}(\lambda) + r(\theta^{(k)},\lambda))V_0^{(k)2}(\theta^{(k)},\lambda)] \| V_0^{(k)} \|^{-2} \\ &\theta^{(k)} = \arg\min_j \left|1-\theta_j^{(k)}\right|, \quad \theta_j^{(k)}(\lambda) = \operatorname{Arg} \Delta^{(k)}(\theta,\lambda) \\ &\left|\mu^{(k+1)}(\lambda) - \mu(\lambda)\right| \leqslant C_\varepsilon \varepsilon^{(k)2}(\lambda), \quad \left|\varepsilon^{(k)}\right| \leqslant C^{-1}(\varepsilon C)^{\eta(k)}, \quad \eta(k) = 2^k \\ &\left|V_0^{(k)} - U\right| + \left|\dot V_0^{(k)} - \dot U\right| \leqslant C_U \varepsilon^{(k)}, \quad C_U \sim 1, \quad C = C_\varepsilon C_\mu \sim 1, \quad |\varepsilon| \ll 1 \end{aligned} \tag{2.8}$$

Algorithm (2.1), (2.2), (2.6) and (2.8) possesses the property of accelerated (quadratic) convergence with respect to the parameter ε. Two or three iterations with respect to k enable us to construct a virtually exact solution $\mu(\lambda)$, $U(t, \mu, \lambda)$ of auxiliary problem (1.6) when $|\varepsilon C| \sim 0.1$.

Hence, using the effective numerical–analytical method of accelerated convergence we have constructed a solution of the classical periodic boundary-value problem (1.6) for any admissible value of λ. The recursive algorithm involves successive refinement of the quantity $\mu^{(k+1)}(\lambda)$ (2.8) using the known estimate of $\mu^{(k)}(\lambda)$, by numerical or analytical integration of the two Cauchy problems (2.1) $V_{1,2}^{(k+1)}$ and the determination of the root $\theta^{(k+1)}(\lambda)$ closest to $\theta = 1$. In addition to integration and calculation of the solution of boundary-value problem $V_0^{(k+1)}(t, \lambda)$ it is required to calculate the square of the norm. This operation can be carried out by a highly accurate method of numerical integration, for example, Simpson's method or when integrating the Cauchy problems, calculating the determinant and determining the quantity $\theta^{(k+1)}(\lambda)$. It is well known [10] that the square of the norm $\|V_0\|^2$ can be represented in a finite form using the function $W(t, \lambda)$

$$\|V_0\|^2 \equiv \int_0^\theta V_0^2 dt = \dot{V}_0(\theta,\lambda)W(\theta,\lambda) - V_0(\theta,\lambda)\dot{W}(\theta,\lambda) \tag{2.9}$$

$$\ddot{W} + [\mu(\lambda) + r(t,\lambda)]W = -V_0(t,\lambda), \quad W(0) = \dot{W}(0) = 0$$

We will now consider the original generalized problem (1.1), the solution of which can be obtained by a numerical or numerical-graphical method by solving auxiliary problem (1.6) for a sufficiently dense set of values of $\lambda \in \Lambda$.

3. THE SOLUTION OF THE GENERALIZED PERIODIC PROBLEM

To construct the required function $\mu(\lambda)$, $\lambda \in \Lambda$ we will use the method of continuation with respect to a parameter. We will use the value of μ obtained as the initial approximation μ^0 for $\lambda + \delta\lambda$, where the addition $\delta\lambda$ is sufficiently small, which is established as a result of the numerical experiment from the condition for the algorithm described above to converge. The sign of the addition $\delta\lambda$ is sufficiently small, which is established as a result of the numerical experiment from the condition for the algorithm described above to converge. The sign of the addition $\delta\lambda$ should be chosen so as to reduce $|\mu|$

$$\operatorname{sign}\delta\lambda = -\operatorname{sign}\mu'(\lambda)\mu(\lambda), \quad \mu'(\lambda) = \frac{1}{\|U\|^2}\int_0^1 r'_\lambda(t,\lambda)U^2 dt \tag{3.1}$$

The value and sign of the derivative $\mu'(\lambda)$ in (3.1) are found using the known solution for the previously specified value of λ. The procedure of continuation with respect to the parameter λ is carried out until $\mu(\lambda) = 0$ and the corresponding λ is determined. It is natural to assume that $\mu'(\lambda) \neq 0$ in the neighbourhood of the values of λ under consideration. When $\mu'(\lambda) = 0$ an additional investigation is required to analyse the behaviour of the function $\mu(\lambda)$ and to determine the possibility of other zeros to exist, i.e. other eigenvalues λ.

As a result we obtain a numerical–analytical method of constructing the required solution λ, $\mu(t)$ of the original generalized periodic boundary-value problem (1.1) by solving the family of solutions $\mu(\lambda)$, $U(t, \mu, \lambda)$ of classical periodic boundary-value problems of the Hill problem type.

Note that it is best to take, as the initial approximation, $\mu^0(\lambda + \delta\lambda)$, a quantity which leads to an error $O(\delta\gamma^2)$

$$\mu^0(\lambda + \delta\lambda) = \mu(\lambda) + \mu'(\lambda)\delta\lambda, \quad \operatorname{sign}(\mu'\delta\lambda) = -\operatorname{sign}\mu \tag{3.2}$$

where the quantity $\mu'(\lambda)$ is found from (3.1).

Using the scheme described above for refining $\mu(\lambda)$, expression (3.1) for $\mu'(\lambda)$ and the approximate relation (3.2), we can implement the procedure for refining the required value of λ, based on the recursive formula ($k = 0, 1, 2, \ldots$)

$$\lambda^{(k+1)} = \lambda^{(k)} - \left[\varepsilon^{(k)}\left(\lambda^{(k)}\right)M^{(k)}\left(\theta^{(k)}, \lambda^{(k)}\right) + \mu^{(k)}\left(\lambda^{(k)}\right)\right] / \mu^{(k)\prime}(\lambda^{(k)}) \tag{3.3}$$

$$\lambda^{(0)} = \lambda^0, \quad \mu(\lambda^0) = O(\varepsilon); \quad \mu^{(k)}(\lambda^{(k)}) = O(\varepsilon^{(k)})$$

Here λ^0 is the known initial approximation $\lambda \in \Lambda$, chosen from additional considerations using variational or other methods (see above) and which ensures the ε-smallness of the quantity $\mu(\lambda^0)$. By

(3.3) the accelerated convergence $\lambda^{(k)} \to \lambda$, similar to (2.8), occurs, which enables the required solution λ, $\mu(t)$ of problem (1.1) to be constructed effectively.

It should, however, be noted that the scheme described above for constructing a solution of the generalized problem may lead to excessive calculations, connected with the construction of the auxiliary function $\mu(\lambda)$ over a possibly extremely wide range of admissible values of $\lambda \in \Lambda$. This procedure can be simplified considerably if one uses the method of accelerated convergence for a direct refinement of a certain estimate λ^0, which leads to a small value of the parameter ε

$$\varepsilon = 1 - \theta^0, \quad |\varepsilon| \ll 1, \quad \theta^0 = \arg\min_j |1 - \theta_j^0|, \quad \theta_j^0 = \operatorname*{Arg}_{\theta} \Delta(\theta, \lambda^0) \tag{3.4}$$

$$\Delta(t, \lambda^0) = \dot{\upsilon}_1(t, \lambda^0) + \upsilon_2(t, \lambda^0) - 2$$

$$\ddot{\upsilon} + r(t, \lambda^0)\upsilon = 0; \quad 1)\, \upsilon(0) = 0, \ \dot{\upsilon}(0) = 1; \quad 2)\, \upsilon(0) = 1, \ \dot{\upsilon}(0) = 0$$

The solution $\upsilon_0(t, \lambda^0)$, which satisfies the periodicity conditions in the interval $0 \leqslant t \leqslant \theta^0$ in the case of a simple root θ^0, is determined using the particular solutions υ_1 and υ_2 (3.4) similar to (2.6). In the case of a multiple root the solution can have the form $\upsilon_0 = C_1\upsilon_1 + C_2\upsilon_2$ (see above). The value of λ is refined using the following recursive scheme

$$\lambda^{(k+1)} = \lambda^{(k)} + \varepsilon^{(k)} L(\theta^{(k)}, \lambda^{(k)}), \quad \theta^{(k)} = \theta(\lambda^{(k)}) = \arg\min_j |1 - \theta_j^{(k)}|$$

$$\theta_j^{(k)} = \theta_j(\lambda^{(k)}) = \operatorname*{Arg}_{\theta} \Delta(\theta, \lambda^{(k)}), \quad \varepsilon^{(k)} = \varepsilon(\lambda^{(k)}) = 1 - \theta^{(k)}$$

$$L(\theta^{(k)}, \lambda^{(k)}) = -[r(\theta^{(k)}, \lambda^{(k)})\upsilon_0^2(\theta^{(k)}, \lambda^{(k)}) + \dot{\upsilon}_0^2(\theta^{(k)}, \lambda^{(k)})] / N(\theta^{(k)}, \lambda^{(k)}) \tag{3.5}$$

$$N(\theta, \lambda) = \int_0^\theta r_\lambda'(t, \lambda)\upsilon_0^2(t, \lambda)dt = \dot{\upsilon}_0(\theta, \lambda)w(\theta, \lambda) - \upsilon_0(\theta, \lambda)\dot{w}(\theta, \lambda)$$

$$\ddot{w} + r(t, \lambda)w = -r_\lambda'(t, \lambda)\upsilon_0, \quad w(0) = \dot{w}(0) = 0, \quad N \neq 0$$

It is assumed that the "square of the norm" N (3.5) (see expression (1.5)) of the function υ_0 with weight r_λ' in the interval $0 \leqslant t \leqslant \theta$ is non-zero. In the classical case when $r(t, \lambda) \equiv \lambda\rho(t) + q(t)$, where $\rho(t) \geqslant \rho_0 > 0$, a standard expression exists for the square of the norm with positive weight $\rho(t)$. Note also that in the case of a multiple root $\theta^0 = \theta(\lambda^0)$ splitting of the eigenvalue of the problem λ in a small neighbourhood of $\lambda = \lambda^0$ is possible.

With the above assumptions the recursive method (3.4), (3.5) possesses the property of accelerated (quadratic) convergence with respect to the small parameter ε. In combination with the procedure of continuation with respect to a parameter of the generalized problem (1.1) it enables one to solve a number of interesting problems of the theory of oscillations, stability and mathematical physics with high efficiency. The algorithm can be tested by the numerical solution of model problems, for example, of the form (1.2) and (1.3) for which an analytical solution is known [10].

For practical calculations the following observation is extremely useful. If the function $r(t, \lambda)$ is even in t, it follows from the identity $u(t) \equiv u^+(t) + u^-(t)$, where the function $u^+(t) = (u(t) + u(-t))/2$ is an even solution, and the function $u(-t) = (u(t) - u(-t))/2$ is an odd solution, that the periodic boundary-value problem (1.1) can be split into two generalized Sturm–Liouville-type problems [11, 12], i.e. we have

$$r(t, \lambda) \equiv r(-t, \lambda); \quad 1)\, u(0) = u(1) = 0; \quad 2)\, \dot{u}(0) = \dot{u}(1) = 0 \tag{3.6}$$

A similar assertion holds for auxiliary classical problem (1.6).

4. SPATIAL OSCILLATIONS OF A DYNAMICALLY SYMMETRICAL ARTIFICIAL SATELLITE

Following the approach used in [12], we will consider the perturbed plane non-linear oscillations of an axisymmetrical artificial satellite (a rigid body), moving in a circular orbit. The equatorial axis remains approximately orthogonal to the plane of the orbit and the polar axis performs small angular oscillations about this plane. These oscillations are described, in the linear approximation, by a Hill-type equation with a periodic coefficient [12]

$$q'' + f(\psi, p_\psi, \alpha, k)\omega^{-2}(\alpha, k)q = 0, \quad f \equiv (p_\psi + 1)^2 - 3(\alpha - 1)\sin^2\psi \tag{4.1}$$

Here the primes denote derivatives with respect to the argument of the angular variable w, with respect to which the function f is 2π-periodic. The angular coordinate ψ and the momentum p_ψ represent these plane oscillations of the polar axis. They are expressed using Jacobi elliptic functions sn and cn and the complete elliptic integral of the first kind K [13, 14]

$$\psi = \arcsin\{k\,\mathrm{sn}(\upsilon, k)\}, \quad \upsilon = (2/\pi)\mathbf{K}(k)w, \quad p_\psi = k\sqrt{3(\alpha-1)}\,\mathrm{cn}(\upsilon, k) \tag{4.2}$$

$$\omega = (\pi/2)(3(\alpha-1))^{1/2}/\mathbf{K}(k), \quad k^2 = 2h_0(3(\alpha-1))^{-1}, \quad 0 \leqslant k < 1$$

Here α ($1 \leqslant \alpha \leqslant 2$) is the ratio of the polar and equatorial moments of inertia in the case of an "oblate" ellipsoid; in the case of a "prolate" ellipsoid ($0 \leqslant \alpha \leqslant 1$) in (4.2) we make the replacement $\psi \to \psi + \pi/2$. As in [12], we will confine ourselves to investigating the first case. Further, the modulus k of the elliptic functions is defined in terms of the constant energy h_0 of the satellite oscillations, which must be fairly small: $2h_0 < 3(\alpha - 1)$. If $2h > 3(\alpha - 1)$, the body performs rotations [12], which can be investigated in a similar way. According to (4.2) the modulus k defines the amplitude ψ_0 of plane oscillations: $\psi_0 = \arcsin k$, $0 \leqslant \psi_0 < \pi/2$.

Hence, Eq. (4.1) contains two independent parameters α and h_0. For applications it is of interest to construct periodic solutions, i.e. to determine the values of the parameters which admit of the existence of such solutions. From the mechanical point of view, the parameters α and ψ_0 are clearer. It is required to determine the relations $\alpha(\psi_0)$ (or $\psi_0(\alpha)$), for which periodic solutions exist [12].

To use the method of accelerated convergence described in Section 3, it is more convenient to introduce the parameters $\lambda = (3(\alpha - 1))^{-1/2}$ and k, taking into account the constraints $\lambda \geqslant 3^{-1/2} \approx 0.577$, $0 \leqslant k < 1$ and the dimensionless time $t = w/2\pi$, with respect to which the coefficient of Eq. (4.1) has a period of unity. As a result of these transformations, we arrive at the following generalized periodic problem of the form (1.1)

$$\ddot{q} + 16\mathbf{K}^2(k)[(\lambda + k\cos\phi)^2 - k^2\sin^2\phi]q = 0, \quad q(t+1) \equiv q(t) \tag{4.3}$$

$$\dot{\phi} = 4\mathbf{K}(k)\sqrt{1 - k^2\sin^2\phi}, \quad \phi(0) = 0, \quad 3^{-1/2} \leqslant \lambda < \infty, \quad 0 \leqslant k < 1$$

Note that, for small values of k, when we can neglect terms $O(k^2)$, the first equation of (4.3) can be reduced to the form of a Mathieu equation ($\lambda^2 \to \lambda$, $\lambda k \to \gamma$). Independent numerical integration of the equation for ϕ enables the calculation of the coefficient of q to be simplified considerably; it is found in terms of Jacobi elliptic functions, i.e. trigonometric series [13, 14]. Further, it is preferable to calculate the complete elliptic integral of the first kind $K(k)$ when $0 \leqslant k^2 \leqslant 0.99$ using quadratures, in accordance with the definition, and when $0.99 < k^2 < 1$ it is extremely convenient to use the asymptotic form [14]

$$\mathbf{K}(k) = \varkappa + \frac{1}{4}(\varkappa - 1)k'^2 + \frac{9}{64}\left(\varkappa - \frac{7}{6}\right)k'^4 + \frac{25}{256}\left(\varkappa - \frac{37}{30}\right)k'^6 + O(k'^8), \tag{4.4}$$

$$\varkappa = \ln(4/k'), \quad k'^2 = 1 - k^2$$

Representation (4.4) leads to a small error of the order $k'^8 \approx 10^{-8}$ when $k'^2 \leqslant 10^{-2}$. Note that Eq. (4.3) as $k^2 \to 1$ possesses a singularity, since $\mathbf{K}(k) \to \infty$. This case requires an asymptotic analysis (see below).

Since the coefficient $r(t, \lambda, k)$ in front of q in Eq. (4.3) is a symmetrical function of t, by (3.6) the construction of the solution of the generalized periodic problem reduces to two generalized Sturm–Liouville-type problems: (1) problems of the first kind for an odd function of q^- and (2) problems of the second kind for an even function of q^+, i.e.

$$1)\ q^-(0) = q^-(1) = 0, \quad \lambda = \lambda_n^-(k), \quad \lambda_n^-(0) = n/2$$

$$q_n^-(t,k) = q^-(t, \lambda_n(k), k), \quad q_n^-(t,0) = (n\pi)^{-1}\sin n\pi t \tag{4.5}$$

$$2)\ \dot{q}^+(0) = \dot{q}^+(1) = 0, \quad \lambda = \lambda_n^+(k), \quad \lambda_n^+(0) = n/2$$

$$q_n^+(t,k) = q^+(t, \lambda_n(k), k), \quad q_n^+(t,0) = \cos n\pi t, \quad n = 1, 2, \ldots$$

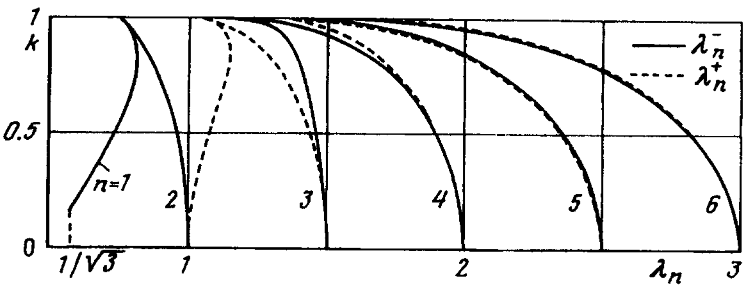

Fig. 1.

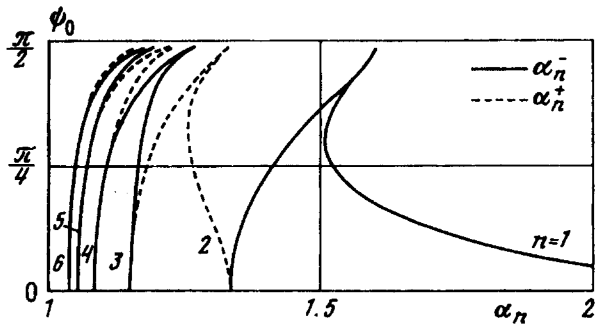

Fig. 2.

For values of $k > 0$ $(0 \leqslant k^2 \leqslant 0.999)$ we use the procedure of continuation with respect to the parameter k and the scheme of the method of accelerated convergence, described in Section 3. The coefficient N (3.5) is calculated by simultaneous integration of the equation $N = r'_\lambda(t, \lambda, k)q^2$ for each n and for each branch of the solutions indicated in (4.5). The corresponding highly accurate calculations of $\lambda_n^\mp(k)$ are shown in Fig. 1 for $n = 1, 2, \ldots, 6$. It should be noted that $\lambda_1^+(k) \leqslant 1/2$. This leads to physically unrealizable values of $\lambda_1^+(\alpha_1^+ > 2)$. The part of the branch $\lambda_1^-(k)$ with $0 \leqslant k \leqslant 0.15$ also leads to physically unrealizable values of $\lambda_1^-(\alpha_1^- > 2)$, but for $k > 0.15$ the values of $\lambda_1^+(k) > 0.577$, i.e. $\alpha_1^- < 2$. All the remaining branches of the resonance curve $\lambda_n^\mp$, $n \geqslant 2$ turn out to be physically realizable $(1 \leqslant \alpha_n^\mp < 2)$ and have a corresponding order of tangency when $k = 0$.

From the theoretical and applied points of view the asymptotic form $\lambda_n^\mp(k)$ as $k \to 1$ is of interest. From the form of the coefficient $r(t, \lambda, k)$ as $k \to 1$ and the properties of elliptic functions (cn $\to 0$, sn $\to 1$, $0 < t < 1/2$, $1/2 < t < 1$), taking (4.4) into account, we obtain that $\lambda_n^\mp(k) \to 1$. Unlike the classical curves (see, for example, the Ince–Strutt diagrams [1–3] and the Haupt diagrams [1]) the even and odd curves alternate in pairs.

The resonance curves of $\alpha_n^\mp(\psi_0)$ for the parameters α and ψ_0 which have a clearer mechanical interpretation are shown in Fig. 2. They are obtained by simple recalculation using the formulae $\alpha_n = 1 + (3\lambda_n^2)^{-1}$, $\psi_0 = \arcsin k$. It is worth presenting both sets of resonance curves (Figs 1 and 2). It is interesting to note that the resonance curve $\alpha_1^-(\psi_0)$ when $\psi_0 \geqslant 0.16$ cuts off a considerable part of the domain of admissible values of the parameters α and ψ_0. It follows from the above analysis that $\alpha_n^\mp(\psi_0) \to 4/3$ as $\psi_0 \to \pi/2$. The curves of α_n^- and α_n^+ also alternate in pairs.

The problem of the stability and instability of the oscillations of the satellite with respect to spatial perturbations both in the linear approximation and in the framework of the complete non-linear model [12], requires further detailed investigation. Linear oscillations can be effectively analysed using the method of accelerated convergence described in Section 3 and methods of the theory of the stability of motion of Hamiltonian systems.

The resonance curves $\lambda(k)$ or $\alpha(\psi_0)$ in the case of rotations of the satellite $(2h_0 > 3(\alpha - 1))$ can be calculated in a similar way. The results of the calculations presented above confirm the undoubted high efficiency of the method of accelerated convergence for solving complex generalized periodic problems, which cannot easily be solved by other familiar approaches.

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